

Different Non-extensive Models for heavy-ion collisions

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Abstract

The transverse momentum (p_T) spectra from heavy-ion collisions at intermediate momenta are described by non-extensive statistical models. Assuming a fixed relative variance of the temperature fluctuating event by event or alternatively a fixed mean multiplicity in a negative binomial distribution (NBD), two different linear relations emerge between the temperature, T , and the Tsallis parameter $q - 1$. Our results qualitatively agree with that of G. Wilk. Furthermore we revisit the "Soft+Hard" model, proposed recently by G. G. Barnaföldi *et.al.*, by a T -independent average p_T^2 assumption. Finally we compare results with those predicted by another deformed distribution, using Kaniadakis' κ parametrization.

Keywords: non-extensive; heavy-ion collisions; Tsallis parameter q ; negative binomial distributions; Kaniadakis' kappa-distribution

1. Introduction

Non-extensive statistics, as the simplest and most natural generalization of the canonical Boltzmann–Gibbs statistics, can be applied widely. Heavy-ion collisions also offer an excellent field for such applications. The transverse momentum spectra exhibit power-law-tailed behavior, that can be described with non-extensive distribution functions better than with Boltzmann distributions,

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$\exp(-\beta E) \rightarrow \exp_q(-\beta E)$. Here the notion of q -exponential, $e_q(x)$, stands for

$$e_q(x) := [1 + (1 - q)x]^{\frac{1}{1-q}}. \quad (1)$$

This form was first suggested by V. Pareto in 1896 [1] for describing the distribution of wealth, and it has been recently promoted by C. Tsallis [2] in connection with non-extensive entropy. The use of a cut power-law in high energy physics was first introduced by R. Hagedorn [3], describing the data of the invariant cross section of hadrons as a function of p_T over a wide range.

In order to gain an improved insight into this generalized non-extensive statistics, as well as to value its application to p_T spectra on the top of pQCD motivated power-laws, ever used for high-energy jets, the meaning of its key parameter, q , must inevitably be studied. Considering the thermodynamical properties of hot and dense nuclear matter formed in heavy-ion collisions, on the other hand it is worth to study the fluctuations of the relevant physical observables, with a large number of particles produced. For example the multiplicity fluctuations in heavy-ion collisions have been investigated by several authors, including G. Wilk *et al.*[4]. Connecting to such studies, we have obtained Tsallis' distribution for particular particle number fluctuation patterns due to a finite heat bath, for details see [5].

On the other hand, further non-extensive distribution functions were being proposed and studied to describe the data on p_T spectra in heavy-ion collisions. K. Urmosy *et al.* proposed a model in which the hadrons produced in heavy-ion collisions stem from a quark-gluon plasma (QGP), referred to as 'soft' yields, on top of those stemming from jets, called 'hard' yields [6]. Both of these contributions are distributed according to respective cut power-laws, $\exp_q(-\beta E)$. Based on this view in this paper we investigate the scatter of the parameter q with respect to the fitted temperatures T . This investigation should differentiate between hard and soft origin of power laws and should look different for heavy-ion as for pp data.

Considering quantum-statistics even further constraints arise from particle-hole CPT symmetry. They lead to a special requirement on the cut power-law

function, alternatively called deformed exponential [7]:

$$e_q(x) \cdot e_q(-x) = 1. \quad (2)$$

For the Pareto-Hagedorn-Tsallis distribution, $\exp_q(x)$, it is easy to see that it does not satisfy this relation, therefore it is inadequate to apply naively extended quantum statistical distributions in the form

$$n_{B,F} = \frac{1}{e_q(x) \mp 1} \quad (3)$$

with $x = (\omega - \mu)/T$, where ω , μ and T are the energy, chemical potential and temperature of the system, respectively, following the notation used in [8]. In the present paper we also shall apply the κ -exponential distribution, proposed by G. Kaniadakis for the description of relativistic plasmas [9]. We compare our corresponding analysis with the ones using Tsallis' distribution, discussed above. The κ -exponential function has the advantage that it readily satisfies the particle-hole symmetry demand:

$$e_\kappa(x) \cdot e_\kappa(-x) = 1, \quad (4)$$

based on the definition

$$e_\kappa(x) := [\sqrt{1 + (\kappa x)^2} + \kappa x]^{\frac{1}{\kappa}}. \quad (5)$$

2. p_T spectra with finite heat capacity

Particle transverse momentum (p_T) spectra provide a tool for measuring thermal properties of the QGP formed in ultra-relativistic heavy-ion collisions. Nowadays more and more experimental data show that the p_T spectrum exhibits power-like rather than the earlier expected exponential behavior, and multiparticle distributions are also broader than expected [10]. The number of particles in such processes is not that large ($N \sim 10^3 - 10^5 \ll 10^{23}$, much less than Avogadro's number). All of these warrant a suitable modification of the thermal model by accounting for possible intrinsic fluctuations in observables measured in high-energy collisions.

The non-extensive form of statistical mechanics, proposed by Tsallis [2], has found applications in many fields including high energy physics [11, 12]. In several situations the p_T spectra can be best described by a Tsallis distribution, characterized by a non-extensive parameter q and a scale parameter T [13]. G. Wilk *et al.* have demonstrated that q turns to be a function of fluctuations of temperature T , together with fluctuations of other variables, from an analysis of different experimental data [14]. Moreover, considering the statistical power-law tailed distributions as canonical distributions in a thermal system connected to a heat reservoir with finite heat capacity[5], we have the combined expression

$$q = 1 - \frac{1}{C} + \frac{\Delta T^2}{T^2}, \quad (6)$$

as next to leading order approximation in the total size of the system. Here C denotes the heat capacity of the reservoir, and $1/C = dT/dE$. It is connected with the temperature and the underlying equation of state, $S(E)$, of the system via the relations $\langle S'(E) \rangle = 1/T$ and $\langle S''(E) \rangle = -1/CT^2$. Here the averaging is done over the ensemble of different collision events with fluctuating parameters. For temperature fluctuations with the Gaussian variance, $\Delta T/T = 1/\sqrt{C}$, following from traditional consideration, one arrives at $q = 1$: nothing but the classical Boltzmann–Gibbs case.

Based on this more generalized function of the Tsallis parameter q , we consider two different situations:

1. Constant Relative Variance due to $\beta = S'(E)$ -Fluctuations. In multi-particle production processes in heavy-ion collisions, it has been shown that the local temperature, $1/\beta$, fluctuates from point to point around some equilibrium value, $T = \langle 1/\beta \rangle$. However, for different collisions the relative variance due to its fluctuations stays approximately constant, namely

$$\sigma^2 = \frac{\Delta T^2}{T^2} \approx \text{const.} \quad (7)$$

Thus we conclude from Eq.(6) that

$$T = E[\sigma^2 - (q - 1)] \quad (8)$$

where we approximate the finite heat capacity by the ideal gas ratio $C = E/T$.

2. Constant Average Occupancy. For elementary pp collisions on the other hand, the measurements of charged particle multiplicity distributions are well described by the negative binomial distribution (NBD) for the full rapidity region.[15] Within the NBD scenario we have the probability

$$P_n = \binom{n+k}{n} f^n (1+f)^{-n-k-1} \quad (9)$$

therefore we obtain the generating function

$$G(t) = \sum P_n e^{nt} = -(k+1) \ln(1+f-f e^t). \quad (10)$$

For the expectation value and variance in the number we get

$$\begin{aligned} \langle n \rangle &= G'(0) = (k+1)f \\ \Delta n^2 &= G''(0) = (k+1)f(f+1). \end{aligned} \quad (11)$$

Assuming now that the average occupancy of phase space by the newly produced hadrons, f , is constant, we arrive at another relation between T and q :

$$T = \frac{E}{f}(q-1). \quad (12)$$

Again we used relations valid for ideal gases, $C = E/T = \langle n \rangle$ and $\Delta T^2/T^2 = \Delta n^2/\langle n \rangle^2$.

In Fig.1 the comparison between our model predictions and G. Wilk's collection is shown. From it we conclude that (i) For the fitting parameters with a Tsallis distribution in AA collisions, assuming constant relative variance of temperature for different collisions works well with experimental data. (ii) For pp collisions the relative fluctuations are no longer the same but the average occupancy parameter in the NBD looks constant for the full rapidity region. This makes another line of correlation between T and q as seen in Fig.1. There is a remarkable agreement between our simple model predictions and G. Wilk's extensive data fits.

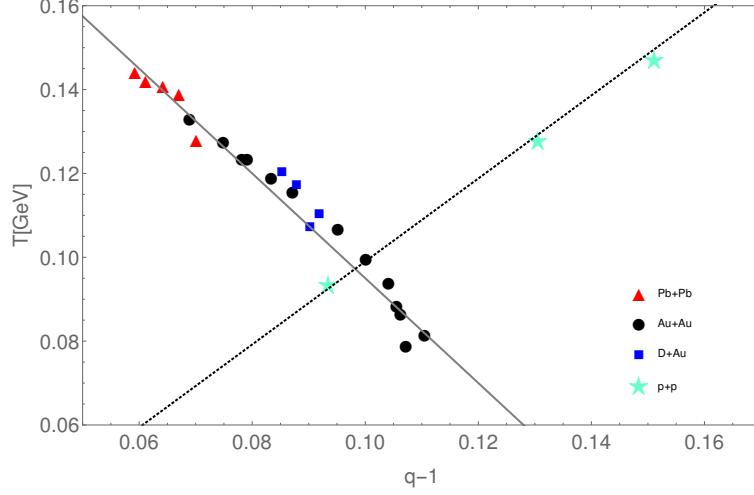


Figure 1: (Color Online) Data are from G. Wilk's collection in his talk given at the Erice School on Complexity, 2015[14]. our fitting plot is shown with the black line, $T = 0.22 - 1.25(q - 1)$ GeV for the constant relative variance σ^2 case ($E = 1.25$ and $\sigma^2 = 0.176$) following Eq.(8) and the dotted one $T = q - 1$ GeV for the constant average occupancy f case ($E/f = 0.99$) from Eq.(12), respectively.

3. p_T spectra within A 'Soft+Hard' Model

Recently a so-called 'soft+hard' model was proposed in Ref.[6]. According to this model the p_T spectra are divided into hadron yields stemming from a supposedly 3d-isotropic QGP (soft) and from quasi 1d-jets (hard), respectively:

$$p^0 \frac{dN}{d^3p} = (p^0 \frac{dN}{d^3p})_{hard} + (p^0 \frac{dN}{d^3p})_{soft}. \quad (13)$$

For both yields the classical Tsallis distribution, $e_q(-\omega/T)$, has been applied with independently fitted T and q parameters [16].

We apply in the present work a radial blast wave flow based Doppler factor to the heavy-ion data, interpreting the spectral temperature as

$$T := T_D \sqrt{\frac{1-v}{1+v}}, \quad (14)$$

with T_D Doppler-shifted Temperature. This is a simplification compared to the classical Tsallis distribution, that interprets the single particle energy as a

relativistic expression with a given mass, m :

$$\omega := \gamma(m_T - vp_T) - m, \quad (15)$$

with $m_T = \sqrt{m^2 + p_T^2}$ transverse mass and $\gamma = 1/\sqrt{1 - v^2}$. For $p_T \gg m$ these expressions agree. Then we inspect the p_T spectra of charged hadrons as a sum of hard and soft yields

$$p^0 \frac{dN}{d^3p} = \sum_i C_i \cdot e_{q_i}(-\omega_i/T_i) \quad (16)$$

for $i = \text{soft}, \text{hard}$ and the C_i -s being normalization constants.

For light particles like the pion in the observed p_T range the assumption $p_T \gg m$ can safely be made and therefore $\omega \rightarrow \gamma(1 - v)p_T$ leads to

$$\frac{\omega}{T} \rightarrow \frac{\gamma(1 - v)p_T}{T} = \frac{p_T}{T_D}. \quad (17)$$

Investigating pion yields from various centrality classes we fit therefore a sum of soft and hard q -deformed distributions using the variable $x = p_T/T_D$:

$$p^0 \frac{dN}{d^3p} \Big|_i = C_i \cdot e_{q_i}(-p_T/T_{Di}). \quad (18)$$

Moreover, for p_T spectra stemming from the same identified hadrons but just for different centralities, the average p_T^2 can also be considered as a constant; it is connected with nothing but the collision energy. In this model approximation we obtain

$$\langle p_T^2 \rangle \approx \frac{\int p_T^3 dp_T (1 + \frac{q-1}{T_D} p_T)^{-\frac{1}{q-1}}}{\int p_T dp_T (1 + \frac{q-1}{T_D} p_T)^{-\frac{1}{q-1}}} = \frac{6 T_D^2}{(4 - 3q)(5 - 4q)}. \quad (19)$$

In order to test these assumptions we fitted p_T spectra measured in $PbPb$ collisions at $\sqrt{s} = 2.76$ TeV by the ALICE experiment using the functional form in Eq.(16). We used identified hadron spectra and then the fitted parameters were compared with our results from above, cf. Fig.2. In Fig.2, we present the 'soft + hard' model fits for the p_T spectra of charged pions in $PbPb$ collisions at $\sqrt{s} = 2.76$ TeV, as seen in [16]. The fitted parameter values are listed in Tab.1; here fluctuations and normalization factors are not explicitly shown, for

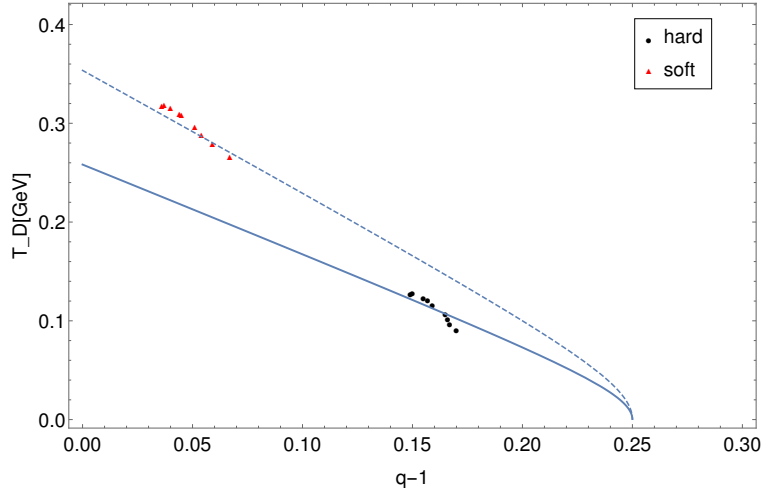


Figure 2: Parameters are from fitting the Transverse momentum spectra of charged pions stemming from different centralities of $PbPb$ collisions at $\sqrt{s} = 2.76$ TeV, as well as our fittings with the 'soft + hard' model using Tsallis- q deformed distributions Eq.(18), which are compared with our theoretical results, and curves are fits of constant $\langle p_T^2 \rangle$ values via Eq.(19).

simplicity . As for the fitting parameters, we used our theoretical and phenomenological model with the constant average p_T^2 to describe the connection between T_{Di} and q_i . Moreover, it is demonstrated that parameters fit the theoretical results quite well. In this fit for the soft part, we obtain $\langle p_T^2 \rangle \approx 0.75$ GeV^2 , while $\langle p_T^2 \rangle \approx 0.40$ GeV^2 for the hard part. This also supports that p_T spectra can be studied in two parts, like in the 'soft + hard' model [6].

In the followings we demonstrate that using Kaniadakis' kappa-distribution, fit for the non-extensive extension of quantum statistics, works equally well as the Tsallis one in the 'soft + hard' model, cf. Fig.3. The spectra using the κ -deformed distribution Eq.(5) are given as

$$p^0 \frac{dN}{d^3p} \Big|_i = C_i \cdot e_{\kappa_i}(-p_T/T_{Di}) \quad (20)$$

Then the corresponding $\langle p_T^2 \rangle$ is

$$\langle p_T^2 \rangle = \frac{6 T_D^2}{1 - 16\kappa^2}. \quad (21)$$

In Fig.3 it is shown that this model also works quite well with κ -deformed distri-

Table 1: Fit parameters to p_T spectra in $PbPb$ collisions within the 'soft + hard' model of Tsallis- q .

Centrality,	Parameters:	q_{soft}	$T_{D,soft}$	q_{hard}	$T_{D,hard}$
0 ~ 5		1.036	0.317	1.170	0.091
5 ~ 10		1.037	0.318	1.167	0.097
10 ~ 20		1.040	0.315	1.166	0.102
20 ~ 30		1.044	0.309	1.165	0.107
30 ~ 40		1.045	0.308	1.159	0.116
40 ~ 50		1.051	0.296	1.157	0.121
50 ~ 60		1.054	0.288	1.155	0.124
60 ~ 70		1.059	0.279	1.150	0.129
70 ~ 80		1.067	0.266	1.149	0.128

butions. Within the constant $\langle p_T^2 \rangle$ the fitting parameters κ_i and T_{Di} are related with each other too, their values are listed in Tab.2. Worth to be mentioned that the total sum of soft and hard contributions to $\langle p_T^2 \rangle$ should be and is indeed model-independent: $0.75 + 0.40 \approx 0.88 + 0.24$.

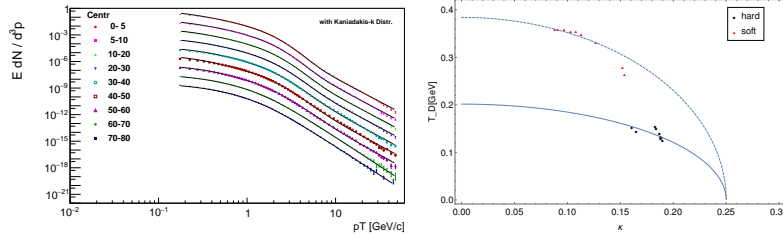


Figure 3: Left panel: Transverse momentum spectra of charged pions stemming from different centrality $PbPb$ collisions at $\sqrt{s} = 2.76 TeV$, as well as our fittings with the 'soft + hard' model using κ - deformed distributions Eq.(20). Right panel: Fit parameters are compared with our theoretical results: the curves are fits of Eq.(21) with the respective $\langle p_T^2 \rangle$ values of $0.88 GeV^2$ and $0.24 GeV^2$ for the soft and hard part.

Table 2: Fit parameters of p_T spectra in $PbPb$ collisions with the 'soft + hard' model using Kaniadakis' κ .

Centrality, Parameters	κ_{soft}	$T_{D,soft}$	κ_{hard}	$T_{D,hard}$
0 \sim 5	0.088	0.358	0.190	0.126
5 \sim 10	0.091	0.359	0.188	0.134
10 \sim 20	0.097	0.359	0.189	0.132
20 \sim 30	0.103	0.355	0.187	0.141
30 \sim 40	0.108	0.354	0.184	0.150
40 \sim 50	0.113	0.347	0.188	0.129
50 \sim 60	0.127	0.330	0.183	0.154
60 \sim 70	0.152	0.279	0.161	0.153
70 \sim 80	0.154	0.264	0.165	0.145

4. Conclusions and outlook

In order to interpret the transverse momentum spectrum in heavy-ion collisions in terms of thermal concepts, we have considered several models based on non-extensive statistical mechanics. Both the original and a generalized Tsallis-Pareto distributions were fitted. Firstly, as QGP is a well known candidate for a thermal system, we have reminded that Tsallis' q parameter is connected to the finite heat capacity of the reservoir as well as to the intrinsic fluctuations of temperature. Then we have demonstrated that for pp and AA collisions the non-extensive parameter q and temperature T must fulfill different relations. Different formulas have been derived and both results are well fitted with other phenomenological results, like G. Wilk's data collection shown in a talk at the Erice School on Complexity, in 2015 [14].

As a further approach we have considered a constant $\langle p_T^2 \rangle$ for various centrality collisions, and used this model to investigate the collection of fit parameters q and T . Based on experimental data taken at different centralities in $PbPb$

collisions at $\sqrt{s} = 2.76$ TeV, we fitted the 'soft + hard' model to p_T spectra. The Tsallis' q is found to be well connected with the temperature, T , gathering along lines following from simple arguments.

We have also investigated this correspondence using another deformed distribution, Kaniadakis' κ -distribution. We have demonstrated that the 'soft+hard' model works equally well with this deformed exponential function. This is important, because quantum statistical distributions, like the Fermi and Bose distributions, can be constructed by using this particle-hole symmetric deformation of the exponential easily. In the future we plan to explore the applicability of the κ -distribution to detailed spectra stemming from ultra-relativistic heavy-ion collisions.

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